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Caustics in general relativity II. The WKB approximation

Y Manor

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge, CB3 9EW, UK

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Abstract. The first-order WKB approximation to solutions of Einstein's equations near a conjugate point (with respect to a fixed space-like surface) is studied. It is found that the metric can be expressed in terms of a generalized Airy function. The geometric meaning of the approximate solution is discussed and it is shown that it is completely characterized by the shear of the two null hypersurfaces defined at a conjugate point. Applications to the question of gravitational energy flux at a conjugate point are pointed out.

1. Introduction

One of the well known results of geometrical optics (see, for example, Landau and Lifshitz 1971) is the eikonal equation

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}=0 \tag{1}$$

for the phase ϕ of the vector potential (or of the field itself). The null hypersurfaces $\phi=$ constant form the electromagnetic wavefronts and the light rays are null geodesics orthogonal to these hypersurfaces. Thus at a focal point, where neighbouring light rays approach each other to first order, there exist several different wavefronts, or, the concept of a wavefront is not well defined at such points. This situation was studied by Airy (1838) almost 140 years ago and it was found (see Ludwig 1966) that near a focal point one has to replace the usual expression for the vector potential in geometrical optics which is

$$A_\mu=a_\mu(x)\exp(\mathrm{i}\omega\phi(x)) \tag{2}$$

by the more complicated form

$$A_\mu=a_\mu(x)\exp(\mathrm{i}\omega\phi(x))A(-\omega^{2/3}u)+\mathrm{i}\omega^{-1/3}b_\mu(x)\exp(\mathrm{i}\omega\phi(x))A'(-\omega^{2/3}u)+\mathrm{O}(\omega^{-2}) \tag{3}$$

where A is the Airy function (i.e., an everywhere regular solution of the differential equation $\mathrm{d}^2/\mathrm{d}x^2[F(x)]=xF(x)$, A' is the derivative of A, u is a smooth function on space–time and ω is a positive parameter (the frequency of the electromagnetic radiation) which is supposed to be much larger than one.

In the general theory of relativity one can study rapidly oscillating, approximate solutions of Einstein's equations (Isaacson 1968, Choquet-Bruhat 1969, MacCallum and Taub 1973) which are the analogue of geometrical optics in Maxwell's theory. One assumes, similarly to equation (2), that the metric is of the form

$$g_{\mu\nu}(x,\omega)=g_{(0)\mu\nu}(x)+\omega^{-1}g_{(1)\mu\nu}(x)\cos(\omega\phi)+O(\omega^{-2}). \tag{4}$$

Let us recall how one arrives at equation (4): we form the expression $\exp(\mathrm{i}\omega\phi)$ where ϕ is the phase function of the geodesics. We suppose that the metric $g_{\mu\nu}(x,\omega)$ can be developed in a power series in ω^{-1} and we multiply all the ω-dependent terms in this series by $\exp(\mathrm{i}\omega\phi)$. Consequently we get the following equation:

$$\begin{aligned} g_{\mu\nu}(x,\omega) &= g_{(0)\mu\nu}(x)+\mathrm{Re}\left(\exp(\mathrm{i}\omega\phi)\sum_{n=1}^{\infty}\omega^{-n}g_{(n)\mu\nu}(x)\right) \\ &= g_{(0)\mu\nu}(x)+\mathrm{Re}[\exp(\mathrm{i}\omega\phi)\omega^{-1}g_{(1)\mu\nu}(x)]+O(\omega^{-2}) \end{aligned} \tag{5}$$

where Re denotes the real part. We demand that the tensors $g_{(n)\mu\nu}(x)$ will be real and we arrive at equation (4). As in electrodynamics one sees, upon substituting the metric (4) in the field equations, that the gravitational wavefront is a null hypersurface and the rays, null geodesics, are orthogonal to this hypersurface. Again, at conjugate points the approximation to the metric given by equation (4) is invalid because the phase function ϕ becomes singular there. It was rigorously shown by Landau and Lifshitz (1971) and by Hawking and Ellis (1973) (see also Boyer 1964) that Einstein's equations imply the existence of conjugate points in the vicinity of matter. It follows that in order to be able to use the rapidly oscillating approximation of the metric near matter (and away from the singularities) we have to find the analogue of equation (3). In the present work we shall find this analogue and we shall construct the first-order WKB solution of Einstein's equations near a conjugate point, paying special attention to the geometric meaning of the results.

From the construction of the approximate metric (4) it is clear that at a conjugate point we have to replace the phase function ϕ by another, non-singular function. To this end we shall use our results (Manor 1976, 1977) concerning the phase function of the geodesics at a conjugate point. Suppose that we are interested in the conjugate points formed by the geodesics originating perpendicularly from a fixed space-like surface. We have shown (Manor 1977) that in the vicinity of every point in space–time there exists an everywhere regular function Ψ of four variables such that the following points are relevant. (i) The four variables of Ψ are chosen among the coordinates x^{μ} of the geodesics and their momenta p_{λ} in such a way that the Poisson brackets of any two of these variables vanish. (ii) Away from conjugate points the function Ψ reduces to the ordinary phase function (i.e., the Hamilton–Jacobi function of the geodesics) ϕ. (iii) If the four variables of Ψ are $x^{\alpha}, p_b (\alpha, b=0,\ldots,3$ where α takes those values which b does not take) then the other four coordinates of the conjugate point (in phase space) (x,p) are given by

$$\frac{\partial\Psi}{\partial x^{\alpha}}=-p_{\alpha} \qquad \frac{\partial\Psi}{\partial p_b}=x^{b}. \tag{6}$$

(iv) If the geodesics under consideration are time-like and the space-like surface to which they are orthogonal is three-dimensional then there exist coordinates (x^{λ},p_{μ}) near the conjugate point in which the function Ψ takes the form

$$\Psi(p_0,x^1,x^2,x^3)=\tfrac{1}{3}p_0^3-p_1x^1-p_2x^2-p_3x^3 \tag{7}$$

while if the geodesics are null and the space-like surface is two-dimensional the function Ψ takes the form

$$\Psi(x^0, \ldots, x^{3-d}, p_{3-d+1}, \ldots, p_3)$$
$$= \tfrac{1}{2}p_0 p_{3-d+1}^2 + \ldots + \tfrac{1}{2}p_0 p_3^2 - p_0 x^0 - \ldots - p_{3-d} x^{3-d} \tag{8}$$

where $d (d = 1, 2)$ is the multiplicity of the conjugate point, that is, the number of linearly independent Jacobi fields vanishing at this point.

Using these results we shall construct, in § 2, the analogue of the Airy function in general relativity. In § 3 we shall find the first-order WKB solution of the field equations at a conjugate point, and in § 4 we shall discuss the geometric meaning of the solution and its bearings upon the question of gravitational energy flux at a conjugate point.

2. The Airy function in general relativity

The immediate generalization of equation (4) to the case in which there is a conjugate point would be to use the new phase function Ψ (given by equation (7) or (8)) instead of the classical function ϕ. But here we encounter two difficulties. (a) the function Ψ depends on the phase space coordinates. In particular, near a conjugate point it depends explicitly on the momenta p_μ. If we substituted Ψ instead of ϕ in equation (4) we would find that the right-hand side is p_μ dependent while the left-hand side is not. (b) the forms given in equations (7) and (8) refer to a particular coordinate system. Therefore, if we use them our results will be manifestly coordinate dependent whereas they should be covariant.

Let us now show how we overcome the first difficulty. Let us first treat the case in which the projection of the phase space point (x, p) on space–time is outside a vicinity of the conjugate points. As remarked in Manor (1977), we can then find coordinates (x^λ, p_μ) centred at (x, p) in which Ψ is linear in x and in p, that is,

$$\Psi(x, p) = -p_\lambda x^\lambda. \tag{9}$$

We put Ψ, as given in equation (9), instead of ϕ in equation (5) and we allow the tensors $g_{(n)\mu\nu}(x)$ to be functions of the momenta p_λ. We then obtain

$$\tilde{g}_{\mu\nu}(x, p, \omega) = \tilde{g}_{(0)\mu\nu}(x, p) + \mathrm{Re}\left(\exp(\mathrm{i}\omega\Psi(x, p)) \sum_{n=1}^{\infty} \omega^{-n} \tilde{g}_{(n)\mu\nu}(x, p)\right). \tag{10}$$

The fact that ω^{-1} appears only in integer powers in the right-hand side of equation (10) is inessential, and one can also consider expressions such as

$$\tilde{g}_{\mu\nu}(x, p, \omega) = \tilde{g}_{(0)\mu\nu}(x, p) + \mathrm{Re}\left(\exp(\mathrm{i}\omega\Psi(x, p)) \sum_{n=1}^{\infty} \omega^{-n+\delta} \tilde{g}_{(n)\mu\nu}(x, p)\right) \tag{11}$$

where δ is a fixed real number. It is easy to see (cf Manor 1977) that the mapping $p_\lambda \to p_\lambda + h_{,\lambda}$ where h is a smooth function on space–time serves as a gauge transformation. Under such transformations equation (9) takes the form

$$\Psi(x, p) = -p_\lambda x^\lambda + \zeta(x). \tag{12}$$

Putting Ψ as given in equation (12) into equation (11) we get

$$\tilde{g}_{\mu\nu}(x,p,\omega)=\tilde{g}_{(0)\mu\nu}(x,p)+\mathrm{Re}[\exp(\mathrm{i}\omega\zeta(x))\exp(-\mathrm{i}\omega p_\lambda x^\lambda)\omega^{-1+\delta}\tilde{g}_{(1)\mu\nu}(x,p)]+\mathrm{O}(\omega^{-2+\delta}). \tag{13}$$

Now we simply integrate equation (13) over the momenta p_λ. But we have to remember that expression (9) is valid only in a vicinity of the point (x, p), that is, equation (13) is valid only in this neighbourhood and the overall averaging is therefore meaningless. In order to take care of this locality we multiply both sides of equation (13) by a smooth function $\chi(p)$ which vanishes outside some neighbourhood of the origin. (Recall that the coordinate system (x^λ, p_μ) is centred at (x, p).) Now we can integrate over the p_μ (where each p_μ runs from $-\infty$ to $+\infty$), divide by $\int\chi(p)\,\mathrm{d}^4p$ and obtain

$$g_{\mu\nu}(x,\omega)=g_{(0)\mu\nu}(x)+\mathrm{Re}\Big(\exp(\mathrm{i}\omega\zeta(x))\omega^{-1+\delta}\int\exp(-\mathrm{i}\omega p_\lambda x^\lambda)\tilde{g}_{(1)\mu\nu}(x,p)\chi(p)\,\mathrm{d}^4p\Big)$$
$$+\mathrm{O}(\omega^{-2+\delta}). \tag{14}$$

The integral in equation (14) is just the Fourier transform of the function $\tilde{g}_{(1)\mu\nu}(\omega^{-1}x,p)\chi(p)$. If we assume that this transform is real and that $\delta=0$ then we shall re-establish equation (4) with $\zeta(x)$ instead of $\phi(x)$.

The discussion beginning at equation (9) proposes a general way to overcome the difficulty due to the fact that the function Ψ depends explicitly on the coordinates p_λ. (Difficulty (*a*) at the beginning of this section.) We simply have to average over all momenta p_λ with a weight function $\chi(p)$ satisfying rather mild demands. The solution of difficulty (*b*), namely, the fact that expressions (7) and (8) are valid only in a particular coordinate system, is very natural. After the averaging over the momenta we replace all the coordinates x^λ by arbitrary functions of the coordinates (which is, in fact, nothing but a gauge transformation, cf Manor 1977).

We shall exhibit this process for $\Psi(x,p)$ given by equation (8). By a simple coordinate transformation in the (x, p) phase space we transform Ψ to the form

$$\Psi(x^0,\ldots,x^{3-d},p_{3-d+1},\ldots,p_3)=\tfrac{1}{2}p_0p_{3-d+1}^2+\ldots+\tfrac{1}{2}p_0p_3^2-\tfrac{1}{2}dp_0^3-p_\lambda x^\lambda \tag{15}$$

satisfying the equation

$$\frac{\partial\Psi}{\partial p_i}=0 \qquad 3-d+1\leqslant i\leqslant 3. \tag{16}$$

Equation (15), which is easier to work with than equation (8), turns Ψ into one of the Thom's elementary catastrophes (see Manor 1977). For simplicity we shall treat the case $d=1$ and for convenience we shall invert the sign of the coordinate x^0. As a result of averaging over the momenta, the metric takes the form

$$g_{\mu\nu}(x,\omega)=g_{(0)\mu\nu}(x)+\mathrm{Re}\Big(\omega^{-1+\delta}\int\exp(\mathrm{i}\omega\phi_3)\exp(-\mathrm{i}\omega p_1x^1-\mathrm{i}\omega p_2x^2)$$
$$\times g_{(1)\mu\nu}(x,p)\chi(p)\,\mathrm{d}^4p\Big)+\mathrm{O}(\omega^{-2+\delta}) \tag{17}$$

with

$$\phi_3(x,p)=\tfrac{1}{2}p_0p_3^2-\tfrac{1}{2}p_0^3+p_0x^0-p_3x^3. \tag{18}$$

As in the example treated above we shall use the gauge transformation $p_\lambda\to p_\lambda+h_{,\lambda}$ where this time h is a function of x^2 and x^3 only, integrate over p_1 and p_2 (i.e., Fourier transform) and redefine the coordinates x^2 and x^3 such that $\tilde{x}^2=\omega x^2$ and $\tilde{x}^3=\omega x^3$. In the new coordinates the metric (17) can be written as

$$g_{\mu\nu}(x,\omega)=g_{(0)\mu\nu}(x)$$
$$+\mathrm{Re}\left(\omega^{-1+\delta}\exp(\mathrm{i}\omega\theta(x))\int\exp(\mathrm{i}\omega\phi_3)g_{(1)\mu\nu}(x,p_0,p_3)\chi(p_0,p_3)\,\mathrm{d}p_0\,\mathrm{d}p_3\right)$$
$$+\mathrm{O}(\omega^{-2+\delta}) \qquad (19)$$

where $\theta(x)$ is some smooth function of the coordinates x^λ. Now we choose $\delta=\frac{2}{3}$. Recall that away from conjugate points we chose $\delta=0$. For Ψ of the form (8) (or (15)) we take $\delta=\frac{2}{3}$ and for Ψ of the form (7) we take $\delta=\frac{1}{3}$. In other words, 3δ is the number of 'non-trivial' integrations on the variables p_μ. Thus we see that the parameter δ characterizes the point around which we build the WKB approximation to Einstein's equations.

The function $g_{(1)\mu\nu}(x,p_0,p_3)\chi(p_0,p_3)$ is a smooth function of p_0,p_3 vanishing outside some neighbourhood of the origin of the p_λ coordinates. We can develop this function in a Taylor series around the origin,

$$g_{(1)\mu\nu}(x,p_0,p_3)\chi(p_0,p_3)$$
$$=k_{\mu\nu}(x)+p_0l_{\mu\nu}(x)+p_3q_{\mu\nu}(x)+p_0^2r_{\mu\nu}(x)+p_0p_3s_{\mu\nu}(x)+p_3^2t_{\mu\nu}(x)+\mathrm{O}(p^3). \qquad (20)$$

We substitute (20) in (19) and perform the integration. In terms of the function

$$F=F(y,z)=4\pi 2^{1/3}3^{-1/3}\int_0^\infty\cos(p_3z)A(-2^{1/3}3^{-1/3}(\tfrac{1}{2}p_3^2+y))\,\mathrm{d}p_3 \qquad (21)$$

where A is the Airy function, $y=\omega^{2/3}x^0$ and $z=\omega^{2/3}x^3$, the metric can be written as

$$g_{\mu\nu}(x,\omega)=g_{(0)\mu\nu}(x)+\mathrm{Re}\left[\exp(\mathrm{i}\omega\theta(x))\left(\omega^{-1}\tilde{a}_{\mu\nu}(x)F+\mathrm{i}\omega^{-1/3}\tilde{b}_{\mu\nu}(x)\frac{\partial F}{\partial y}\right.\right.$$
$$\left.\left.+\mathrm{i}\omega^{-4/3}c_{\mu\nu}(x)\frac{\partial F}{\partial z}+\omega^{-5/3}d_{\mu\nu}(x)\frac{\partial^2F}{\partial z^2}\right)\right]+\mathrm{O}(\omega^{-2}). \qquad (22)$$

Up to this point all the calculations, starting from equation (17), were performed in that coordinate system (x^λ, p_μ) in which the phase function Ψ is given by equation (15). Now we can go over to arbitrary coordinates in space–time. Equation (22) will retain its form except that y and z will become arbitrary smooth functions of the coordinates.

We thus overcame the two difficulties mentioned at the beginning of this section. Equation (22) is the form of the metric for the first-order WKB approximate solutions of Einstein's equations in a vicinity of a conjugate point of the type described by the phase function (15).

The function $F(y,z)$, defined in equation (21) is the general relativistic analogue of the Airy function in electrodynamics, in case the geodesic phase function $\Psi(x,p)$ is given by equation (8) (or (15)). One can absorb all the numerical coefficients in the functions y and z and assume that F is given by

$$F(y,z)=J\int_0^\infty\cos(pz)A(-p^2-y)\,\mathrm{d}p \qquad (23)$$

where J is some number. Note that $F(y, z)$ as given in equation (23), is a special, everywhere-regular solution of the differential equation

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + y\right) k(y, z) = 0. \tag{24}$$

Now F is not completely determined since the functions y and z appearing in it are arbitrary functions of the coordinates. We can therefore choose any everywhere-regular solution of equation (24). We shall use the simple solution

$$k(y, z) = A(-y - \omega^{2/3}) \exp(\mathrm{i}\omega^{1/3} z). \tag{25}$$

Under these circumstances, and assuming that all the tensors $\tilde{a}_{\mu\nu}(x), \ldots, d_{\mu\nu}(x)$ which appear in equation (22) are real, the metric takes the form

$$g_{\mu\nu}(x, \omega) = g_{(0)\mu\nu}(x) + \omega^{-1} \cos(\omega\theta) a_{\mu\nu}(x) A(-\omega^{2/3}(u+1)) + \omega^{-4/3} \sin(\omega\theta) b_{\mu\nu}(x) A'(-\omega^{2/3}(u+1)) + \mathrm{O}(\omega^{-2}) \tag{26}$$

where u is a smooth function of the coordinates. From now on we shall use this simplified form.

Up till now we treated the case in which the phase function $\Psi(x, p)$ is given by equation (15). When $\Psi(x, p)$ is of the form (7) we go through the same steps and find that the WKB form of the metric for this type of conjugate point is

$$g_{\mu\nu}(x, \omega) = g_{(0)\mu\nu}(x) + \omega^{-1} \cos(\omega\theta) a_{\mu\nu}(x) A(-\omega^{2/3} v) + \omega^{-4/3} \sin(\omega\theta) b_{\mu\nu}(x) A'(-\omega^{2/3} v) + \mathrm{O}(\omega^{-2}) \tag{27}$$

where v is a smooth non-negative function of the space–time coordinates. This WKB form of the metric corresponds to the form (3) of the vector potential in geometrical optics near a focal point.

3. The first-order WKB approximate solution

With the metric given by equations (26) or (27) one can construct the first-order WKB approximate solution to Einstein's equations near a conjugate point. In particular we shall deal with the case in which the conjugate point is along null geodesics orthogonal at their starting point to a fixed space-like two-surface. The results in the other case will be given at the end of this section.

Einstein's equations involve not only the metric but also its derivatives and the inverse metric. We shall assume that the inverse metric can be written in a form similar to the metric itself, that is

$$g^{\mu\nu}(x, \omega) = g^{(0)\mu\nu}(x) + \omega^{-1} g^{(1)\mu\nu}(x) + \omega^{-4/3} g^{(2)\mu\nu}(x) + \mathrm{O}(\omega^{-2}) \tag{28}$$

where $g^{(i)\mu\nu}$, $i = 0, 1, 2$ are determined from the algebraic equations

$$g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}_{\lambda}. \tag{29}$$

The derivatives of the metric which is given in equation (26) can be written as follows:

$$g_{\mu\nu,\lambda} = g_{(0)\mu\nu,\lambda} - M^{+} \psi^{+}_{\lambda} m^{-}_{\mu\nu} - M^{-} \psi^{-}_{\lambda} m^{+}_{\mu\nu} + \mathrm{O}(\omega^{-1}) \tag{30}$$

where the functions $\psi^{\pm}_{\lambda}$, $M^{\pm}$ and $m^{\pm}_{\mu\nu}$ are defined as

$$\psi^{\pm}_{\lambda} = [\theta \pm \tfrac{2}{3}(u+1)^{3/2}]_{,\lambda} \equiv \psi^{\pm}_{,\lambda} \tag{31}$$

$$m^{\pm}_{\mu\nu} = a_{\mu\nu} \pm (u+1)^{1/2} b_{\mu\nu} \tag{32}$$

$$M^{\pm} = \tfrac{1}{2}[\sin(\omega\theta)A(-\omega^{2/3}(u+1)) \pm \omega^{-1/3}(u+1)^{-1/2}\cos(\omega\theta)A'(-\omega^{2/3}(u+1))]. \tag{33}$$

(Note that we are interested only in a neighbourhood of the conjugate point, that is, in those values of u which are close to zero. Hence $u+1$ never vanishes.)

We then calculate the Ricci tensor $R_{\mu\nu}$ and equate to zero those terms of $R_{\mu\nu}$ which go to infinity as ω goes to infinity. The result is the following pair of equations:

$$g^{(0)\lambda\rho}(\psi^{+}_{\mu}\psi^{+}_{\lambda}m^{-}_{\rho\nu} + \psi^{+}_{\nu}\psi^{+}_{\rho}m^{-}_{\lambda\mu} - \psi^{+}_{\mu}\psi^{+}_{\nu}m^{-}_{\lambda\rho} - \psi^{+}_{\lambda}\psi^{+}_{\rho}m^{-}_{\mu\nu}) = 0 \tag{34}$$

$$g^{(0)\lambda\rho}(\psi^{-}_{\mu}\psi^{-}_{\lambda}m^{+}_{\rho\nu} + \psi^{-}_{\nu}\psi^{-}_{\rho}m^{+}_{\lambda\mu} - \psi^{-}_{\mu}\psi^{-}_{\nu}m^{+}_{\lambda\rho} - \psi^{-}_{\lambda}\psi^{-}_{\rho}m^{+}_{\mu\nu}) = 0. \tag{35}$$

These are the basic equations of the first-order WKB approximate solutions of Einstein's equations. This type of equation is well known from the study of discontinuities of derivatives of the metric tensor (Stellmacher 1938, Treder 1962). These equations will also appear in calculations of the WKB approximate solutions to Einstein's equations away from conjugate points (Choquet-Bruhat 1969, MacCallum and Taub 1973) if one uses two null hypersurfaces instead of one which is usually done.

The general solution of an equation like (34) is known (Stellmacher 1938). Here we need the general solution of the pair (34)–(35) and it is found as follows: if

$$g^{(0)\mu\nu}\psi^{+}_{\mu}\psi^{+}_{\nu} \neq 0 \qquad \text{and} \qquad g^{(0)\mu\nu}\psi^{-}_{\mu}\psi^{-}_{\nu} \neq 0 \tag{36}$$

then the general solution of (34)–(35) is

$$m^{-}_{\mu\nu} = \psi^{+}_{\mu}a_{\nu} + \psi^{+}_{\nu}a_{\mu}; \qquad m^{+}_{\mu\nu} = \psi^{-}_{\mu}b_{\nu} + \psi^{-}_{\nu}b_{\mu} \tag{37}$$

where a^{μ} and b^{ν} are arbitrary vectors. This solution is of no physical interest because using the coordinate transformation

$$x^{\mu} \to x^{\mu} + \omega^{-2}\sin(\omega\theta)A(-\omega^{2/3}(u+1))V^{\mu} + \omega^{-7/3}\cos(\omega\theta)A'(-\omega^{2/3}(u+1))W^{\mu} \tag{38}$$

where

$$V^{\mu} = \tfrac{1}{2}(a^{\mu} + b^{\mu}); \qquad W^{\mu} = \tfrac{1}{2}(u+1)^{-1/2}(a^{\mu} - b^{\mu}) \tag{39}$$

(one raises and lowers indices with the metric $g_{(0)\mu\nu}$) we get a metric in which both $m^{-}_{\mu\nu}$ and $m^{+}_{\mu\nu}$ do not appear at all. Similarly, if one of the vectors ψ^{+}_{μ} or ψ^{-}_{μ} is not a null vector with respect to the metric $g_{(0)\mu\nu}$ then either $m^{-}_{\mu\nu}$ or $m^{+}_{\mu\nu}$ are devoid of any physical meaning. Here we shall be interested in the case in which both $m^{+}_{\mu\nu}$ and $m^{-}_{\mu\nu}$ cannot be gauged away by a coordinate transformation. Therefore we must have

$$g^{(0)\mu\nu}\psi^{+}_{\mu}\psi^{+}_{\nu} = g^{(0)\mu\nu}\psi^{-}_{\mu}\psi^{-}_{\nu} = 0. \tag{40}$$

Now we construct a null tetrad with respect to the metric $g_{(0)\mu\nu}$ by defining two complex valued null vectors m_{μ} and $\bar{m}_{\mu}$ such that we shall have, in addition to (40),

$$g^{(0)\mu\nu}\psi^{+}_{\mu}m_{\nu} = g^{(0)\mu\nu}\psi^{-}_{\mu}m_{\nu} = g^{(0)\mu\nu}m_{\mu}m_{\nu} = 0; \qquad g^{(0)\mu\nu}m_{\mu}\bar{m}_{\nu} = -1. \tag{41}$$

It is easy to verify by expanding the tensors $m^{\pm}_{\mu\nu}$ in the vectors of the null tetrad we have just defined and substituting this expansion in (34)–(35), that the general solution of the equations (34)–(35) is

$$\begin{aligned} m^-_{\mu\nu} &= Hm_\mu m_\nu + \bar{H}\bar{m}_\mu \bar{m}_\nu + \psi^+_{,\mu} a_\nu + \psi^+_{,\nu} a_\mu \\ m^+_{\mu\nu} &= Im_\mu m_\nu + \bar{I}\bar{m}_\mu \bar{m}_\nu + \psi^-_{,\mu} b_\nu + \psi^-_{,\nu} b_\mu \end{aligned} \tag{42}$$

where H and I are complex valued functions and a^μ, b^ν are arbitrary vectors. We have already seen that the vectors a^μ and b^ν do not have any physical meaning and can therefore be put equal to zero without any loss of generality. Consequently we shall assume that the solution of the pair (34)–(35) is

$$m^-_{\mu\nu} = Hm_\mu m_\nu + \bar{H}\bar{m}_\mu \bar{m}_\nu; \qquad m^+_{\mu\nu} = Im_\mu m_\nu + \bar{I}\bar{m}_\mu \bar{m}_\nu. \tag{43}$$

To summarize, equation (43) gives $a_{\mu\nu}$ and $(u+1)^{1/2} b_{\mu\nu}$ in terms of H and I, and the functions θ and u which appear in the metric (cf equation (26)) are related by equations (31) and (40). This is as much as we can say in general. By imposing additional conditions we can completely determine all the functions appearing in the metric. Before we analyse the geometrical meaning of the scalars H and I (and consequently of the tensors $m^\pm_{\mu\nu}$) let us write down the results in the case where the phase function $\Psi(x, p)$ is given by equation (7). (Recall that the results of this section are based on the assumption that $\Psi(x, p)$ is of the form (8) (or (15).) The basic equations remain (34)–(35) with the solutions (43) except that here

$$\psi^\pm_{,\mu} = [\theta \pm \tfrac{2}{3} r]_{,\mu} \equiv \psi^\pm_{,\lambda} \tag{44}$$

and

$$m^\pm_{\mu\nu} = a_{\mu\nu} \pm r b_{\mu\nu} \tag{45}$$

where $r(x)$ is the smooth function satisfying $r^2 = v$ (cf equation (27)). Observe that in this case the tensors $m^\pm_{\mu\nu}$ will not have a direct geometrical meaning connected to the null hypersurfaces $\psi^\pm = \text{constant}$, because a conjugate point of the type in which the phase function $\Psi(x, p)$ takes, in some coordinates, the form (7) is necessarily a conjugate point along time-like geodesics. It is known that the properties of time-like geodesics are not determined, in general, by those of the null geodesics. Thus, for example, one cannot expect the tensors $m^\pm_{\mu\nu}$ given in equation (45) to be directly linked with the intensity of gravitational radiation at a conjugate point.

4. The geometrical meaning of the solution

In order to find out the geometrical meaning of the functions H and I let us construct a null tetrad with respect to the metric $g_{\mu\nu}$. From the form of the inverse metric (cf equations (28) and (29)) and equation (43) it follows that

$$g^{\mu\nu} \psi^+_{,\mu} \psi^+_{,\nu} = g^{\mu\nu} \psi^-_{,\mu} \psi^-_{,\nu} = 0 \tag{46}$$

that is, the vectors $\psi^+_{,\mu}$ and $\psi^-_{,\mu}$ are null also with respect to the metric $g_{\mu\nu}$. As the two other null (with respect to the metric $g_{\mu\nu}$) complex conjugate vectors M^μ and $\bar{M}^\mu$ we choose

$$M^\mu = m^\mu + \tfrac{1}{2}\omega^{-1}(Q + \omega^{-1/3} R)\bar{m}^\mu \tag{47}$$

where the functions Q and R are defined by

$$
\begin{aligned}
Q &= \tfrac{1}{2}\cos(\omega\theta)A(-\omega^{-2/3}(u+1))(\bar{I}+\bar{H}) \\
R &= \tfrac{1}{2}\sin(\omega\theta)A'(-\omega^{2/3}(u+1))(u+1)^{-1/2}(\bar{I}-\bar{H}).
\end{aligned}
\tag{48}
$$

For convenrence we denote

$$n_\mu \equiv (\psi^{+\lambda}\psi^-_\lambda)^{-1}\psi^-_\mu. \tag{49}$$

The degenerate case $\psi^{+\lambda}\psi^-_\lambda = 0$ is especially simple and therefore we shall not discuss it here. With the functions defined in equation (33) we find that

$$\psi^+_{\mu;\nu} = \psi^+_{\mu|\nu} - \tfrac{1}{2}M^- m^+_{\mu\nu}(\psi^{+\lambda}\psi^-_\lambda) + O(\omega^{-1}) \tag{50}$$

and

$$n_{\mu;\nu} = n_{\mu|\nu} - \tfrac{1}{2}M^+ m^-_{\mu\nu} + O(\omega^{-1}) \tag{51}$$

where a semi-colon (vertical rule) denotes a covariant derivation with respect to the metric $g_{\mu\nu}(g_{(0)\mu\nu})$. Now we use the NP spin coefficients (see Newman and Penrose 1962) formed from the null tetrad $(\psi^+_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ (with the metric $g_{(0)\mu\nu}$) as well as those formed from the null tetrad $(\psi^+_\mu, n_\mu, M_\mu, \bar{M}_\mu)$ (with the metric $g_{\mu\nu}$). As usual, we denote by $\sigma(\sigma_0)$ the shear, with respect to the metric $g_{\mu\nu}(g_{(0)\mu\nu})$, of the null geodesic congruence having the tangent vector ψ^+_μ and by λ and λ_0 the same quantities except that the tangent to the geodesics is n^μ. From equation (50) we obtain

$$\sigma \equiv \psi^+_{\mu;\nu}M^\mu M^\nu = \sigma_0 - \tfrac{1}{2}M^-\bar{I}(\psi^{+\lambda}\psi^-_\lambda) + O(\omega^{-1}) \tag{52}$$

and from equation (51) we obtain

$$\lambda \equiv -n_{\mu;\nu}\bar{M}^\mu\bar{M}^\nu = \lambda_0 + \tfrac{1}{2}M^+ H + O(\omega^{-1}). \tag{53}$$

Thus we see that the functions H and I are essentially characterized by the shear of null hypersurfaces. The tensors $m^\pm_{\mu\nu}$ (or equivalently the tensors $a_{\mu\nu}$ and $b_{\mu\nu}$) are determined (up to a phase function which has no physical meaning and which can always be made equal to zero by rotating the vectors m_μ and M_μ) by a single scalar which is $|H|^2$ for $m^-_{\mu\nu}$ and $|I|^2$ for $m^+_{\mu\nu}$. It follows that the tensors $a_{\mu\nu}$ and $b_{\mu\nu}$ which appear in the metric are essentially determined by the squared moduli of the shear of the two null hypersurfaces $\psi^\pm = \text{constant}$ (cf equation (31)) defined at the conjugate point. We say essentially because we have already remarked that not all of the functions involved u, θ, $a_{\mu\nu}$, $b_{\mu\nu}$ are completely fixed by the general first-order WKB approximate solution (they are, however, determined if initial conditions are imposed) and because the expression for σ or λ, as given in equation (52) or (53), contains, besides I and H, oscillatory functions and the Airy function.

Let us now discuss the applications of these results to the question of gravitational energy flux at a conjugate point. As pointed out by Pirani (1957), Bondi *et al* (1962) and Penrose (1967) the only physically meaningful local concept related to gravitational energy is the gravitational energy flux. Since we have well defined null hypersurfaces it is clear that these will be the gravitational wavefronts. Thus we have a well defined direction of the energy flux and we need to know the magnitude of the flux which is evidently a scalar.

To avoid misunderstanding we would like to emphasize that our solution is valid only at some neighbourhood of the conjugate point. We have shown (cf Manor 1977) that it is impossible to construct a global non-singular phase function $\Psi(x, p)$, hence one is unable to construct a global WKB approximation. Consequently, the regular null hypersurfaces $\psi^{\pm}$ = constant are only locally defined. The physical determination and the geometrical description of the null hypersurfaces $\psi^{\pm}$ = constant is parallel to the case in which there are no conjugate points. We observe (by using the asymptotic form of the Airy function) that $\psi^{\pm}$ is, in fact, the phase of the rapidly oscillating metric given by equation (26) much the same as ϕ is the phase of the oscillating metric given by equation (4). Then we apply the analysis of Pirani (1957), Penrose (1967) and Isaacson (1968).

In the beginning of this section we showed that there is only one type of scalar which can be formed from the first-order WKB approximate solution and this is bilinear combinations of the shear of the null hypersurfaces $\psi^{\pm}$ = constant. Hence we are led to the conclusion that the shear should represent the first-order WKB approximation to the gravitational energy flux. Notice that here we are dealing with conjugate points. A similar conclusion away from conjugate points was reached by Penrose (1967). Also we find it necessary to use the shear of both null hypersurfaces and not just one of them.

In order to find out the concrete expression for the approximate gravitational energy flux at a conjugate point we need some results from differential geometry. Suppose there is some observer with unit velocity field v^{μ}. Let S be the space-like hypersurface formed by all geodesics orthogonal to v^{μ} and let N be a null hypersurface. The intersection of S with N is a space-like two-surface. Draw, in this two-surface a small circle D and observe the generators of N which are orthogonal to D. It is known (Sachs 1961) that if one measures the projection of the null geodesics orthogonal to D on the intersection of N with some other space-like hypersurface S' formed from the geodesics orthogonal to some unit velocity field v'^{μ}, one will find that the circle D has been expanded, rotated and sheared. Hence we see that one of the ways to study the effects of curvature is to examine the distortion of 'shadows' cast by null geodesics.

To express this idea quantitatively let us consider a point in a Riemannian two-surface S_2 and denote by T_x the tangent plane to S_2 at x. In T_x we draw a small circle C_r of radius r. We study the geodesics emanating from x whose tangent vectors at x belong to C_r. On each such geodesic we choose the point in which the affine parameter (which is zero at x) reaches the value 1. This set of points describes a closed contour C in the two-surface S_2. (We assume that S_2 is connected, that is, consists of 'one piece'.) It can be shown that the length of C minus the length of C_r (that is, length $(C) - 2\pi r$) is a measure of the curvature of the two-surface S_2. More precisely, as r goes to zero we have

$$\text{length}\,(C) = 2\pi r - \frac{\pi r^3}{3} G_a(S_2) + o(r^3) \tag{54}$$

where G_a is the Gaussian curvature of S_2 at the point x and length (C) is calculated with respect to the Riemannian metric of S_2. Now we consider the case in which S_2 is imbedded in space–time. In particular, we shall be interested in the case in which there are two null vectors l^{μ} and n^{μ} and S_2 is the intersection of all the geodesics orthogonal to $2^{-1/2}(l^{\mu}+n^{\mu})$ with the null hypersurface whose tangent is n^{μ}. Under these conditions equation (54) generalizes to the following equation:

$$\text{length}\,(C) = 2\pi r - 3^{-1}\pi r^3 G_a(S_2) + 2\,.\,3^{-1}\pi r^3(\mu^2 - |\lambda|^2) + o(r^3) \tag{55}$$

where μ and λ are the coefficients of expansion and shear of the null geodesics having the tangent vector n^{μ}. Since we would like to find the contribution of the rapidly oscillating part of the metric we have to calculate length (C) with the metric $g_{\mu\nu}$ and to subtract the contribution from the metric $g_{(0)\mu\nu}$. Thus

$$\begin{aligned}\text{length}\,(C)-\text{length}_{(0)}\,(C) \\ &= -3^{-1}\pi r^3(G_a(S_2)-G_{a(0)}(S_2))-3^{-1}2\pi r^3(|\lambda|^2-|\lambda_0|^2) \\ &\quad +o(r^3)+O(\omega^{-1}) \qquad (56)\end{aligned}$$

where $\text{length}_{(0)}\,(C)$ and $G_{a(0)}(S_2)$ are the length of C and the Gaussian curvature of S_2 calculated using the metric $g_{(0)\mu\nu}$. From the Gauss–Bonnet theorem (see Kobayashi and Nomizu 1969) it follows that

$$\begin{aligned}\text{area}\,(\mathscr{D})\,\text{length}\,(C)-\text{area}_{(0)}(\mathscr{D})\,\text{length}_{(0)}\,(C) \\ &= \int_{\mathscr{D}}\text{length}\,(C)\,\mathrm{d}S_2-\int_{\mathscr{D}}\text{length}_{(0)}\,(C)\,\mathrm{d}_{(0)}S_2 \\ &= -3^{-1}2\pi r^3\left(\int_{\mathscr{D}}|\lambda|^2\,\mathrm{d}S_2-\int_{\mathscr{D}}|\lambda_0|^2\,\mathrm{d}_{(0)}S_2\right)+o(r^3)+O(\omega^{-1}) \qquad (57)\end{aligned}$$

where $\mathscr{D}$ is the region limited by C and $\mathrm{d}S_2$ ($\mathrm{d}_{(0)}S_2$) is the surface element of S_2 with respect to the projection of the metric $g_{\mu\nu}$ ($g_{(0)\mu\nu}$) on S_2.

Now the term proportional to r^3 on the right-hand side of equation (57) is nothing but Bondi's mass loss (Bondi *et al* 1962). Their calculation is restricted to infinity in the asymptotically flat region of space–time but if one considers only the first-order WKB approximation to it, then the result is valid everywhere (away from singularities and conjugate points; cf Penrose 1967).

Thus we arrive at the conclusion that one of the ways to measure the gravitational energy flux of rapidly oscillating fields at conjugate or non-conjugate points is to measure the change of length of a small closed contour (recall that all our results are valid only in the limit $r\to 0$) placed perpendicularly to the gravitational rays (i.e., the null geodesics).

Note that at a conjugate point, λ in equation (57) is given by equation (53) which contains the Airy function.

As far as we were able to see a direct measurement of the gravitational energy flux in the way proposed here is beyond the reach of the present day experimental precision.

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